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ABSTRACT

For large values of the pressure-gradient parameter β (i.e., highly accelerated flows), the order of the similarity equations governing the laminar boundary layer is reduced by two. Coles has shown that this is a singular perturbation problem for which an inner solution and an outer solution must be obtained. In this paper, inner and outer solutions are presented which include the effects of a power-law temperature-viscosity relation, nonunit Prandtl number, leading-edge sweep, and surface mass transfer. An extremely accurate method is described for estimating skin friction and heat transfer for all positive values of β.

SYMBOLS

a = constant of the inner solutions, $\sigma_2 \cos^2\Lambda/t_s$

c_p = fluid specific heat (taken to be constant across the boundary layer)

f = variable related to streamwise velocity; $f' = u/u_e$

f_w = surface mass transfer parameter, $-[(\rho v_w)/(\rho_w \mu_w u_e)]\sqrt{2\xi}$

g = transverse velocity function, w/w_e

H = total enthalpy, $h + u^2/2$

h = static enthalpy, $c_p T$

I_1 = $(1 - t_s)[I_1(1) - I_1(2)] - (1 - t_w)I_1(3) + I_1(2)$

I_2 = $\int_0^\infty f'(1 - f')d\eta$

I_3 = $\int_0^\infty f'(1 - f'^2)d\eta$

$I_1(1)$ = $\int_0^\infty (1 - g^2)d\eta$

$I_1(2)$ = $\int_0^\infty (1 - f'^2)d\eta$

$I_1(3)$ = $\int_0^\infty (1 - \theta)d\eta$

k = thermal conductivity

M = Mach number

Pr = Prandtl number

p = pressure

R = gas constant

s_0 = transformed velocity function defined by Eq. (24)

T = fluid temperature

T_0 = free-stream stagnation temperature, H_e/c_p

t = transformed similarity variable defined by Eq. (24)

t_s = sweep parameter, $[1 + \{(\gamma - 1)/2\}M_\infty^2 \cos^2 \Lambda]/[1 + \{(\gamma - 1)/2\}M_\infty^2]$

t_w = wall temperature ratio, H_w/H_e or T_w/T_0

U_∞ = free-stream velocity

u = flow velocity in the x-direction

v = flow velocity in the y-direction

w = flow velocity in the z-direction

W = constant defined by Eq. (15)

x = coordinate in the direction of flow

y = coordinate normal to the surface

z = coordinate transverse to the flow direction

β = pressure-gradient parameter defined by the similarity relation

$$u_e \sim \left[(T_e/T_0)\xi^\beta\right]^{1/2}, \text{ i.e. } \beta = \frac{2\xi}{u_e} \frac{du_e}{d\xi} \left(\frac{T_0}{T_e}\right)$$

γ = specific heat ratio

η = similarity variable defined by Eq. (3)

θ = dimensionless total enthalpy $(H - H_w)/(H_e - H_w)$

Λ = angle of sweep between leading edge and the normal to the free-stream direction

λ = density-viscosity ratio $(\rho\mu/\rho_w\mu_w)$

μ = viscosity

ξ = transformed x-coordinate defined by Eq. (4)

ρ = fluid density

σ = hypersonic parameter $(U_\infty^2/2H_e)$

σ_1 = modified hypersonic parameter $(U_\infty^2/2H_e) \cdot (u_e/u_\infty)^2$

σ_2 = modified hypersonic parameter $(U_\infty^2/2H_e)[(u_e/u_\infty)^2 \cos^2 \Lambda + \sin^2 \Lambda]$

ω = exponent in the temperature-viscosity law $\mu \sim T^\omega$

Subscripts

$(\ )_e$ = value at the edge of the boundary layer

$(\ )_w$ = value at the wall

$(\ )_\infty$ = value of the function in the free stream

I. INTRODUCTION

The similarity equations governing the laminar boundary layer are well known. For a thermally and calorically perfect gas of homogeneous composition, there are three:[1]

Streamwise Momentum Equation

$$(\lambda f'')' + ff'' = \beta\left\{f'^2 - \frac{1}{t_s}\left[(1 - t_w)\theta - (1 - t_s)g^2 + t_w\right]\right\} \quad (1)$$

Transverse Momentum Equation

$$(\lambda g')' + fg' = 0$$

Energy Equation

$$\left(\frac{\lambda}{Pr}\,\theta'\right)' + f\theta' = \left\{\frac{2\sigma\lambda}{(1 - t_w)}\left(\frac{1}{Pr} - 1\right)\left[f''f'\left(\frac{u_e}{u_\infty}\right)^2 \cos^2\Lambda + gg'\sin^2\Lambda\right]\right\}' \quad (2)$$

In these equations, the quantities f', θ, and g are the normalized streamwise velocity, total enthalpy, and transverse velocity respectively, and primes denote differentiation with respect to the transformed variable η normal to the surface:

$$\eta = \frac{u_e}{\sqrt{2\xi}}\int_0^y \rho dy \quad (3)$$

where

$$\xi = \int_0^x \rho_w \mu_w u_e dx \quad (4)$$

Other quantities are defined in the list of Symbols.

These equations are applicable only when the boundary layer is self-similar, i.e., when derivatives of f, θ, and g with respect to ξ are zero. Often, however, it is possible to obtain reasonably accurate values for boundary-layer properties (including the skin friction, heat-transfer rate, and displacement thickness) by assuming that these properties are given by the similar solution corresponding to the local values of β, t_w, t_s, etc.

In using the concept of local similarity in highly accelerated flows, it is useful to have solutions of the laminar-boundary-layer equations for large values of β. The mathematical difficulties encountered in the limit $\beta >> 1$ may be illustrated by writing the streamwise momentum equation in the form

$$\frac{1}{\beta}\left[(\lambda f'')' + ff''\right] - \left\{f'^2 - \frac{1}{t_s}\left[(1 - t_w)\theta - (1 - t_s)g^2 + t_w\right]\right\} = 0 \quad (5)$$

In the limit $\beta \to \infty$, Eq. (5) reduces from third to first order:

$$\lim_{\beta \to \infty} f' = \left[\frac{1}{t_s}\left\{(1 - t_w)\theta - (1 - t_s)g^2 + t_w\right\}\right]^{1/2} \quad (6)$$

The transverse momentum equation and the energy equation remain of second order. As originally pointed out by Coles[(2)] and as later elaborated by Beckwith and Cohen,[(3)] this leads to a singular perturbation problem in which the thickness of the velocity layer is of order $\beta^{-1/2}$ with respect to total enthalpy and transverse velocity layers of order unity.*

*Discussion of this problem also appears in an abbreviated version in Lagerstrom's article.[(4)]

The appropriate method of solution is similar to that employed in deriving a uniformly valid approximation to the Navier-Stokes equations in the limit of large Reynolds number (see Kaplun and Lagerstrom,[5] Lagerstrom and Cole,[6] and the recent book by Van Dyke[7]).

The Outer Limit Equations

Since the mathematical justification of this singular perturbation solution (more popularly called an inner- and outer-expansion procedure) has been discussed in detail by Coles,[2] our purposes will be served by a cursory development of the governing equations. The present analysis extends the work of Beckwith and Cohen[3] to include a power-law temperature-viscosity relation, a constant but nonunit Prandtl number, and surface-mass transfer.

Assume that appropriate outer representations of the dependent variables f, θ, g are of the following forms:

$$\left.\begin{aligned} f &= f_0 + \frac{1}{\sqrt{\beta}} f_1 + \ldots.; \\ \theta &= \theta_0 + \frac{1}{\sqrt{\beta}} \theta_1 + \ldots.; \\ g &= g_0 + \frac{1}{\sqrt{\beta}} g_1 + \ldots.. \end{aligned}\right\} \tag{7}$$

Substituting these representations into the momentum and energy equations and dropping all terms of order $\beta^{-1/2}$ and smaller,* the following "outer" equations are obtained:

$$f_0' = \left[\frac{1}{t_s}\left\{(1 - t_w)\theta_0 - (1 - t_s)g_0^2 + t_w\right\}\right]^{1/2} \tag{8}$$

$$\left(\frac{\lambda}{Pr}\theta_0'\right)' + f_0\theta_0' + \left\{\frac{2\lambda\sigma}{(1 - t_w)}\left(1 - \frac{1}{Pr}\right)\left[\frac{f_0'^2}{2}\left(\frac{u_e}{u_\infty}\right)^2 \cos^2\Lambda + \frac{g_0^2}{2}\sin^2\Lambda\right]'\right\}' = 0 \tag{9}$$

$$(\lambda g_0')' + f_0 g_0' = 0 \tag{10}$$

The appropriate boundary conditions are found (a) by requiring that the outer equations satisfy the outer exact boundary conditions, and (b) by exact matching of the inner and outer representations. The results may be written

$$f_0(0) = f_w;\ \theta_0(0) = g_0(0) = 0;\ \theta_0(\infty) = g_0(\infty) = 1 \tag{11}$$

Only one boundary condition on f_0 may be satisfied by the outer equations, because the outer limit equation for f_0 reduces to first order. The exact boundary condition $f \to 1$ as $\eta \to \infty$ is automatically satisfied by Eq. (8). The no-slip condition, $f' \to 0$ as $\eta \to 0$, must be satisfied by an

*The outer limit equations are properly obtained by applying the limit $\beta \to \infty$ to the full equations expressed in outer variables, with η held fixed. This gives identical results to those cited here.

inner solution which is valid in a region of extent $\beta^{-1/2}$ with respect to the scale of the outer solution.

In this approximation, the density-viscosity ratio $\lambda = (\rho\mu/\rho_w\mu_w)$ in the outer layer becomes

$$\lambda = \left[\frac{1}{t_w}\left(\frac{1-\sigma_2}{t_s}\right)\left\{(1-t_w)\theta_0 - (1-t_s)g_0^2 + t_w\right\}\right]^{\omega-1} \qquad (12)$$

where

$$\sigma_2 = \left(\frac{U_\infty^2}{2H_e}\right)\left[\left(\frac{u_e}{u_\infty}\right)^2 \cos^2\Lambda + \sin^2\Lambda\right]$$

and we have assumed that the viscosity is equal to a power function of the temperature, i.e., $\mu \sim T^\omega$. As $\eta \to 0$, $\lambda \to [(1-\sigma_2)/t_s]^{\omega-1}$ because $f_0'(0) = (t_w/t_s)^{1/2}$ and $f_0'(0)$ is nonzero in general.* The energy and transverse momentum equations are coupled through the implicit appearance of both θ_0 and g_0 in Eqs. (9) and (10). If Pr = 1, $\theta_0 = g_0$ and the number of coupled ordinary differential equations is reduced from three to two. Setting both the viscosity-temperature exponent ω and the Prandtl number Pr equal to unity reproduces the equations of Beckwith and Cohen.[3] If $\sigma_2 = 0$ (i.e., the local Mach number is zero), $t_s = 1$ and the transverse momentum and energy equations are again uncoupled.

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*It should be noted that the inner and outer expansion procedure breaks down in the limit $t_w \to 0$, because the outer solution for f_0 satisifies the exact boundary conditions for the complete equations and the inner solution for f' is simply zero.

One interesting case was pointed out by Coles and we extend his result to generalized compressible flow. For $t_w = t_s = 1$, $\Lambda = 0$ so that $\sigma_2 = (U_\infty^2/2H_e)(u_e/u_\infty)^2 \equiv \sigma_1$, $\lambda = (1 - \sigma_1)^{\omega-1}$, and $f_0' = 1$ for all η. In taking this limit with $Pr \neq 1$, the product of $(f_0'^2)'$ and $(1 - t_w)^{-1}$ approaches θ_0' so that the energy equation becomes

$$\theta_0'' + (\eta + f_w)Pr[1 - \sigma_1(1 - Pr)]^{-1} (1 - \sigma_1)^{1-\omega}\, \theta_0' = 0 \tag{13}$$

with the boundary conditions

$$\theta_0(0) = 0; \qquad \theta_0(\infty) = 1 \tag{14}$$

If we define the new variable χ and the constants χ_0 and W by

$$\chi = \frac{\eta}{\sqrt{W}} - \chi_0;$$

$$\chi_0 = - f_w/\sqrt{W}; \tag{15}$$

$$W = \frac{(1 - \sigma_1)^{1-\omega}}{[1 - \sigma_1(1 - Pr)]Pr}$$

then Eqs. (13) and (14) are satisfied by the solution

$$\theta = [\mathrm{erf}(\chi/\sqrt{2}) + \mathrm{erf}\,(\chi_0/\sqrt{2})]/[1 + \mathrm{erf}\,(\chi_0/\sqrt{2})] \tag{16}$$

For $\chi_0 < 0$, we note the identity

$$\mathrm{erf}\,(-\chi) = -\mathrm{erf}\,(\chi) \tag{17}$$

The corresponding solution for g_0 is found from Eq. (16) by setting $Pr = 1$.

Equations (8) to (10) with the boundary conditions of Eq. (11) represent a two-point boundary-value problem for the complete solution of the outer limit equations. The derivatives $\theta_0'(0)$ and $g_0'(0)$, along with the integrals I_2, $I_1(1)$, $I_1(2)$, and $I_1(3)$ evaluated using f_0, θ_0, and g_0 in place of f, θ, and g, are given in Table 1. The integral I_1 is identically zero. Beckwith and Cohen[3] calculated several of these quantities for the special case of $Pr = \omega = 1$, $f_w = 0$, and nonunit values of t_s.

The Inner Limit Equations

Inasmuch as the order of the energy and transverse momentum equations are not reduced in taking the limit $\beta \to \infty$, the outer equations (Eqs. (9) to (11)), represent complete solutions for the total enthalpy and transverse-velocity profiles for large β. The terms f_0 and $(f_0'^2)$, which appear in the outer equations, differ from the exact solutions f and (f'^2) only in a region which is $\beta^{-1/2}$ smaller in extent than the region of applicability of the outer equations. Therefore, the outer equations asymptotically represent the complete solutions for θ and g as $\beta \to \infty$, and in this limit the inner solutions for θ and g are identically zero.

The no-slip condition $f'(0) = 0$ is satisfied by the inner limit equations for f; the inner equations for θ and g, as noted previously, reduce to $\theta = g = 0$. To examine the inner streamwise momentum equation, it is necessary to introduce a new independent variable $\tilde{\eta}$ and an inner representation for f:

$$\tilde{\eta} = \sqrt{\beta}\eta; \; f = f_w + \frac{1}{\sqrt{\beta}}\tilde{f}_0(\tilde{\eta}) + \ldots \tag{18}$$

so that

$$\tilde{f}_0'(\tilde{\eta}) = f_0'(\eta) = \frac{u}{u_e} \tag{19}$$

After rearrangement, the inner equation for $\tilde{f}_0'$ becomes

$$(\lambda\tilde{f}_0'')' + \frac{1}{\beta}\tilde{f}_0\tilde{f}_0'' - (\tilde{f}_0')^2 + \frac{t_w}{t_s} = 0 \tag{20}$$

and dropping terms of order $1/\sqrt{\beta}$ and smaller, the result is

$$(\lambda\tilde{f}_0'')' - (\tilde{f}_0')^2 + \frac{t_w}{t_s} = 0 \tag{21}$$

where

$$\lambda = \left[1 - \frac{\sigma_1 \cos^2 \Lambda}{t_w}(\tilde{f}_0')^2\right]^{\omega-1} \tag{22}$$

The boundary conditions on f_0' are found from the exact boundary conditions at the wall and the matching condition* that the inner and outer representations of f agree in the limit $\sqrt{\beta} \to \infty$ with $\tilde{\eta}$ large but fixed. The following boundary conditions for the inner equations result:

$$\tilde{f}_0(0) = 0;\ \tilde{f}_0'(0) = 0;\ \tilde{f}_0(\infty) = \sqrt{t_w/t_s} \tag{23}$$

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*The matching condition applied here is elaborated by Van Dyke:[7] Inner representation of (outer representation) = outer representation of (inner representation).

Since Eqs. (21) and (22) involve only $\tilde{f}_0'$ and not $\tilde{f}_0$, they represent a well-posed second-order two-point boundary-value problem. A more symmetrical form may be obtained by applying the transformation

$$s_0(t) = \sqrt{\frac{t_s}{t_w}}\,\tilde{f}_0'(\tilde{\eta});\; t = (t_w/t_s)^{1/4}\,\tilde{\eta} \tag{24}$$

Then the problem may be written

$$(\lambda s_0')' - s_0^2 + 1 = 0 \tag{25}$$

$$\lambda = \left[1 - a s_0^2\right]^{\omega - 1} \tag{26}$$

$$s_0(0) = 0;\; s_0(\infty) = 1 \tag{27}$$

where the constant a is given by

$$a = (\sigma_1 \cos^2\Lambda)/t_s \tag{28}$$

Values of $s_0'(0)$ satisfying Eqs. (25) to (27) are given in Table 2 for several values of ω and a. By using a simple transformation suggested by J. Aroesty* the two-point boundary-value problem of Eqs. (25) to (27) is reduced to the numerically simpler problem of a double quadrature from 0 to t. Define a new variable α by the relation

$$s_0' = ds_0/dt \equiv \alpha \tag{29}$$

so that the transformation $(s_0, t) \rightarrow (s_0, \alpha)$ is

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*The RAND Corporation, private communication.

$$\frac{d}{dt} = \alpha \frac{d}{ds_0} \tag{30}$$

and hence

$$\alpha \frac{d}{ds_0} (\lambda\alpha) - s_0^2 + 1 = 0 \tag{31}$$

A subsequent transformation $(s_0,\alpha) \rightarrow (s_0,\tau)$ is defined by

$$\tau = \lambda\alpha \tag{32}$$

and upon multiplying Eq. (31) by λ, we have

$$\tau d\tau + \lambda(s_0)[1 - s_0^2]ds_0 = 0 \tag{33}$$

with the single boundary condition $\tau(s_0 = 1) = 0$. Separating variables and integrating Eq. (33) from $s_0 = 1$ to s_0, we have

$$\tau(s_0) = \left\{2\int_1^{s_0} \left(1 - as^{*2}\right)^{\omega-1} \left(s^{*2} - 1\right)ds^*\right\}^{1/2} \tag{34}$$

A final quadrature of Eq. (34) suffices to compute $t(s_0)$; noting that $\tau = \lambda(ds_0/dt)$ and $s_0(t = 0) = 0$, we obtain

$$t(s_0) = \int_0^{s_0} [\lambda(s^*)/\tau(s^*)]ds^* \tag{35}$$

The quadratures of Eqs. (34) and (35) are rapidly accomplished using standard numerical integration formulas. The skin-friction derivative $s_0'(0)$ is found from the quadrature of Eq. (34); because $\lambda(0) = 0$ for all ω and a,

$$s_0'(0) = \alpha(0) = \tau(s_0 = 0) \tag{36}$$

For the special case of $\omega = 1$, Coles[2] pointed out that an analytic solution of Eq. (25) may be found in the form

$$s_0 = 1 - 3\,\mathrm{sech}^2(t/\sqrt{2} + \tanh^{-1}\sqrt{2/3}) \tag{37}$$

Using either the analytic solution for $\omega = 1$ or the numerical solutions for $\omega \neq 1$, the surface skin friction derivation f_w'' is then found by reversing the previous transformations:

$$\lim_{\beta\to\infty} f_w'' = \sqrt{\beta}\left(\frac{t_w}{t_s}\right)^{3/4} s_0'(0) \tag{38}$$

In general, $s_0'(0)$ depends on the three parameters, σ_2, ω, and t_s; for $\omega = 1$, the value of $s_0'(0)$ is 4/3.

II. DISCUSSION

It is very difficult to obtain exact numerical solutions of the laminar boundary-layer equations for values of β greater than 2. The reason is simply that the singular behavior of $f'(\eta)$ near the wall which exists for $\beta \to \infty$ becomes dominant even for moderate values of β. Conversely, the results obtained for $\beta = \infty$ should be good representations of the behavior of the boundary layer for large but finite values of β. One of the important results that we wish to demonstrate is that, by combining the exact numerical results of Ref. 1 for $\beta \leq 5$ and the limiting solutions for $\beta \to \infty$ given in Tables 1 and 2, it is possible to estimate accurately the skin-friction and heat-transfer derivatives f''_w, g'_w, and θ'_w for all positive values of β.

The skin-friction results for $\beta = \infty$ are displayed in Fig. 1. The influences of the two parameters ω and a on the inner solutions for $s'_0(0)$ are seen to be relatively small. For $0.5 \leq \omega \leq 1.0$ and $0 \leq a \leq 1.0$, the value of $s'_0(0)$ may be found by interpolation to better than 0.25 percent.

The skin friction parameter $f''_w \beta^{-1/2}(t_w/t_s)^{-3/4}$ approaches the limit of $s'_0(0)$ as $\beta \to \infty$. The difficulties that were observed previously(1) in calculating exact solutions for $\beta \geq 2$ imply that this limit is approached very rapidly with increasing β. This supposition is borne out by Figs. 2 to 4, where the skin friction parameter is shown as a function of $\beta^{-1/2}$. The limit parameter $\beta^{-1/2}$ is suggested by the ordering procedure used to obtain the inner and outer equations. Solid lines indicate exact numerical solutions from Ref. 1 and dashed lines are extrapolations.

Figure 2 illustrates the approach of the skin-friction parameter to its asymptotic limit $s_0'(0)$ for different wall temperatures. The limiting value is approached most rapidly for high wall temperatures. This result is to be expected from the behavior of the outer equations. Large values of t_w increase the magnitude of the velocity difference across the inner layer, whereas for $t_w \to 0$, the distinction between the inner and outer layers breaks down and no proper limit is obtained. It has also been found empirically[1] that exact solutions are more difficult to obtain for large t_w.

The approach of the skin-friction parameter to its limiting value with increasing β is illustrated in Fig. 3 for several values of the sweep parameter t_s. From a numerical point of view, the accuracy of the present extrapolation procedure increases with increasing sweep.

It is of interest to examine the behavior of the inner and outer equations with mass injection at the wall. Applications of these results include mass transfer cooling of rocket nozzles (Back and Witte [8]) and blunt hypervelocity vehicles. The outer equations determine the heat transfer derivative θ_w' and they contain the boundary condition $f_0(\eta \to 0) = f_w$. Thus, surface mass transfer ($f_w < 0$) acts to reduce θ_w' and consequently surface heat transfer even in the limit $\beta \to \infty$. On the contrary, the skin friction derivative f_w'' is found from the inner solution for $\tilde{f}'(0)$, Eq. (21), and the solution of this equation is independent of the value of f_w. In highly accelerated flows, therefore, the effects of blowing on skin friction become negligible in the limit $\beta \to \infty$ with f_w fixed.

Figure 4 illustrates the behavior of the skin-friction parameter as a function of the injection parameter f_w. The limit is approached

smoothly with decreasing values of $\beta^{-1/2}$ for all f_w, and accurate estimations of f''_w may be obtained for all β by comparing the exact solutions of Ref. 1 for $\beta \leq 5$ and the inner limit solutions of Fig. 1.

The wall heat transfer is related by the modified Stewartson and Howarth-Dorodnitsyn transformations (Eqs. (3) and (4)) to θ'_w. Inasmuch as θ_0 and g_0 represent the complete solutions for θ and g as $\beta \to \infty$, the values of $\theta'_0(0)$ and $g'_0(0)$ represent the asymptotic limits of θ'_w and g'_w as $\beta \to \infty$. A comparison between our present results and Ref. 1 suggests that the asymptotic values are within a few percent of the exact answers for $\beta \geq 7$. It should be emphasized that θ'_w and g'_w become independent of β as $\beta \to \infty$, demonstrating that the heat transfer predicted by a local similarity analysis is highly accelerated flows approaches a limiting value.

Figure 5 shows the typical behavior of the heat-transfer derivative θ'_w with increasing values of β. The behavior of θ'_w with decreasing values of $\beta^{-1/2}$ is seen to be smooth and (at least for the case of $\sigma_1 = 0$) monotonic. In an earlier paper,[9] we demonstrated that the proper parameter to use in comparing different heat-transfer calculations is $\theta'_w[(1 - t_w)/(t_{aw} - t_w)]$. For $\sigma_1 = 0$ and $t_s = 1$, the adiabatic wall temperature t_{aw} is unity for all Pr so that the heat transfer parameter reduces to θ'_w. Although we have not been able to prove it analytically, it appears that $t_{aw} = 1.0$ for all values of Pr, t_s, ω, and σ in the limit $\beta = \infty$. This is a very surprising result and should be examined further.

The boundary-layer displacement thickness δ^* is defined by the relation

$$\delta^* \equiv \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy = \frac{\sqrt{2\xi}}{\rho_e u_e}\left(\frac{T_0}{T_e}\right)\left[I_1 - \left(\frac{T_e}{T_0}\right) I_2\right] \tag{39}$$

where the integrals I_1 and I_2 are given in the list of Symbols. For $\beta = \infty$, the integral I_1 is identically zero and the integral I_2

$$I_2 = \int_0^\infty f'(1 - f')\, d\eta \tag{40}$$

is recorded in Table 2. In the absence of sweep, the velocity profile is monotonic and $0 < f' < 1$, so that the integral I_2 is positive and the displacement thickness is negative. With sweep ($t_s < 1$), there is often an overshoot in the velocity profile so that $f' > 1$ for some range of η and I_2 becomes negative. With sweep, therefore, the displacement thickness may be either positive or negative, depending upon the particular parameters being considered. Numerical comparison between the calculated results of Ref. 1 for moderate β and the present results for $\beta = \infty$ suggests that δ^* monotonically decreases with increasing β. In the special case when $\beta = \infty$ and $t_w = t_s = 1$, $f' = 1$ and δ^* is of the order of $\beta^{-1/2}$ times the scale of boundary layer displacement thickness for $\beta = 0$.

Table 1

SOLUTION OF THE INNER LIMIT EQUATIONS FOR $\beta \to \infty$

ω	$a^{(*)}$	$s_0'(0)$
0.5	0	1.1547
	0.2025	1.1672
	0.4050	1.1816
	0.4920	1.1885
	1.0000	1.2533
0.75	0	1.1547
	0.2025	1.1609
	0.4050	1.1679
	0.4920	1.1712
	1.0000	1.1991
1.0	all	1.1547

*

$$a = \frac{U_\infty^2}{2H_e} \left(\frac{u_e}{u_\infty}\right)^2 \cdot \frac{\cos^2 \Lambda}{t}$$

Table 2

SOLUTION OF THE OUTER LIMIT EQUATIONS FOR $\beta \to \infty$

$I_1 \equiv 0$

ω	f_w	t_s	σ	$\left(\frac{u_e}{u_\infty}\right)^2$	Pr	t_w	$g'(0)$	$\theta'(0)$	I_2	$I_1(1)$	$I_1(2)$	$I_1(3)$
0.5	0	0.1	0.9	1	0.7	0.4	1.128	0.8388	-2.487	0.8330	-3.856	0.6067
						0.6	1.249	0.9283	-3.383	0.8262	-5.055	0.5952
			1	0	0.7	0.4	1.128	0.8388	-2.487	0.8330	-3.856	0.6067
						0.6	1.249	0.9283	-3.383	0.8262	-5.055	0.5952
				0.5	0.7	0.4	0.9489	0.8077	-3.008	0.9803	-4.641	0.6971
						0.6	1.050	0.8939	-4.056	0.9771	-6.043	0.6876
		0.3333	0.9	1	0.7	0.15	0.4837	0.4421	-0.0764	1.269	-0.1279	0.9456
						0.4	0.6181	0.5650	-0.5016	1.344	-0.9260	0.9786
						0.6	0.6840	0.6253	-0.8623	1.348	-1.529	0.9723
			1	0	0.7	0.15	0.6535	0.5016	-0.0039	0.9708	0.0087	0.7648
						0.4	0.8351	0.6410	-0.3297	1.011	-0.6133	0.7824
						0.6	0.9242	0.7094	-0.6099	1.007	-1.085	0.7734
				0.5	0.7	0.15	0.5495	0.4758	-0.0485	1.129	-0.0761	0.8555
						0.4	0.7022	0.6080	-0.4268	1.189	-0.7898	0.8821
						0.6	0.7772	0.6729	-0.7489	1.190	-1.329	0.8750
		1	0	--	0.7	0.15	0.4965	0.4154	0.3019	1.082	0.7833	0.9215
						0.4	0.6345	0.5309	0.2538	1.140	0.5792	0.9653
						0.6	0.7022	0.5875	0.1795	1.143	0.3866	0.9666
						1.0	0.7979	0.6676	0.0	2.128	0.0	1.954
			0.5	--	0.7	0.15	0.4175	0.3789	0.3310	1.262	0.8588	1.010
						0.4	0.5336	0.4842	0.2783	1.342	0.6350	1.058
						0.6	0.5905	0.5359	0.1968	1.351	0.4239	1.060
						1.0	0.6709	0.6089	0.0	2.342	0.0	2.046

Table 2, cont.

ω	f_w	t_s	σ	$\left(\frac{u_e}{u_\infty}\right)^2$	Pr	t_w	$g'(0)$	$\theta'(0)$	I_2	$I_1(1)$	$I_1(2)$	$I_1(3)$
0.5	0	1	0.9	--	0.7	0.15	0.2792	0.2734	0.4587	1.853	1.190	1.400
						0.4	0.3568	0.3494	0.3856	1.986	0.8800	1.467
						0.6	0.3949	0.3867	0.2726	2.009	0.5874	1.468
						1.0	0.4487	0.4394	0.0	3.007	0.0	2.449
	-0.2	1	0	--	0.7	0.15	0.3766	0.3306	0.3439	1.253	0.9003	1.059
						0.4	0.5129	0.4451	0.2804	1.273	0.6418	1.070
						0.6	0.5801	0.5014	0.1962	1.261	0.4232	1.058
						1.0	0.6751	0.5811	0.0	2.228	0.0	2.030
			0.5	--	0.7	0.15	0.3319	0.3080	0.3727	1.431	0.9748	1.147
						0.4	0.4469	0.4126	0.3047	1.473	0.6972	1.162
						0.6	0.5035	0.4640	0.2134	1.468	0.4603	1.151
						1.0	0.5836	0.5367	0.0	2.441	0.0	2.122
			0.9	--	0.7	0.15	0.2403	0.2360	0.4995	2.018	1.303	1.533
						0.4	0.3176	0.3118	0.4116	2.115	0.9411	1.568
						0.6	0.3555	0.3489	0.2890	2.123	0.6232	1.558
						1.0	0.4092	0.0795	0.0	3.104	0.0	2.524
	-0.4	1	0	--	0.7	0.15	0.2730	0.2551	0.3937	1.460	1.041	1.225
						0.4	0.4037	0.3664	0.3106	1.426	0.7134	1.189
						0.6	0.4689	0.4216	0.2149	1.394	0.4644	1.161
						1.0	0.5619	0.5001	0.0	2.338	0.0	2.114
			0.5	--	0.7	0.15	0.2557	0.2441	0.4214	1.630	1.113	1.309
						0.4	0.3675	0.3463	0.3344	1.621	0.7676	1.279
						0.6	0.4230	0.3969	0.2318	1.597	0.5008	1.252
						1.0	0.5019	0.4687	0.0	2.548	0.0	2.205

Table 2, cont.

ω	f_w	t_s	σ	$\left(\frac{u_e}{u_\infty}\right)^2$	Pr	t_w	g'(0)	θ'(0)	I_2	$I_1(1)$	$I_1(2)$	$I_1(3)$
0.5	-0.4	1	0.9	--	0.7	0.15	0.2040	0.2012	0.5451	2.202	1.431	1.684
						0.4	0.2804	0.2760	0.4399	2.255	1.008	1.680
						0.6	0.3181	0.3130	0.3066	2.246	0.6619	1.655
						1.0	0.3714	0.3652	0.0	3.207	0.0	2.604
0.7	0	0.1	0.9	1	0.7	0.15	0.6877	0.5268	-1.516	0.9729	-2.505	0.7355
						0.4	1.131	0.8412	-2.449	0.8233	-3.780	0.6049
						0.6	1.235	0.9182	-3.274	0.7998	-4.865	0.5833
			1	0	0.7	0.15	0.9269	0.6899	-1.347	0.8417	-2.231	0.6287
						0.4	1.131	0.8412	-2.449	0.8233	-3.780	0.6049
						0.6	1.235	0.9182	-3.274	0.7998	-4.865	0.5833
				0.5	0.7	0.15	0.8395	0.7152	-1.572	0.9169	-2.576	0.6679
						0.4	1.021	0.8698	-2.771	0.9058	-4.256	0.6494
						0.6	1.114	0.9486	-3.668	0.8835	-5.435	0.6290
		0.3333	0.9	1	0.7	0.15	0.5855	0.5355	-0.0812	1.209	-0.1396	0.8938
						0.4	0.7091	0.6483	-0.4520	1.211	-0.8341	1.050
						0.6	0.7719	0.7055	-0.7596	1.189	-1.345	0.8608
			1	0	0.7	0.15	0.6966	0.5357	-0.0101	1.036	-0.0036	0.8116
						0.4	0.8468	0.6500	-0.3338	1.024	-0.6207	0.7929
						0.6	0.9231	0.7081	-0.6048	0.9998	-1.075	0.7707
				0.5	0.7	0.15	0.6308	0.5467	-0.0559	1.129	-0.0920	0.8497
						0.4	0.7649	0.6623	-0.4041	1.126	-0.7475	0.8358
						0.6	0.8329	0.7209	-0.6938	1.104	-1.230	0.8147
		1	0	--	0.7	0.15	0.5450	0.4598	0.3313	1.180	0.8471	0.9966
						0.4	0.6582	0.5530	0.2637	1.185	0.6000	1.0000
						0.6	0.7153	0.5999	0.1831	1.168	0.3941	0.9853
						1.0	0.7979	0.6676	0.0	2.128	0.0	1.954

Table 2, cont.

ω	f_w	t_s	σ	$\left(\frac{u_e}{u_\infty}\right)^2$	Pr	t_w	$g'(0)$	$\theta'(0)$	I_2	$I_1(1)$	$I_1(2)$	$I_1(3)$
0.7	0	1	0.5	--	0.7	0.15	0.4931	0.4495	0.3389	1.289	0.8666	1.019
						0.4	0.5943	0.5406	0.2698	1.304	0.6138	0.8082
						0.6	0.6454	0.5865	0.1873	1.289	0.4032	1.008
						1.0	0.7191	0.6526	0.0	2.252	0.0	1.976
			0.9	--	0.7	0.15	0.3887	0.3810	0.3998	1.618	1.022	1.203
						0.4	0.4677	0.4582	0.3183	1.647	0.7242	1.207
						0.6	0.5075	0.4971	0.2210	1.634	0.4757	1.189
						1.0	0.5649	0.5531	0.0	2.594	0.0	2.151
	-0.2	1	0	--	0.7	0.15	0.4227	0.3729	0.3714	1.345	0.9575	1.126
						0.4	0.5355	0.4663	0.2898	1.317	0.6613	1.102
						0.6	0.5926	0.5133	0.1996	1.285	0.4304	1.076
						1.0	0.6751	0.5811	0.0	2.228	0.0	2.030
			0.5	--	0.7	0.15	0.3929	0.3663	0.3789	1.452	0.9768	1.149
						0.4	0.4941	0.4576	0.2958	1.435	0.6750	1.125
						0.6	0.5452	0.5036	0.2038	1.406	0.4394	1.098
						1.0	0.6190	0.5699	0.0	2.351	0.0	2.052
			0.9	--	0.7	0.15	0.3261	0.3208	0.4394	1.778	1.131	1.331
						0.4	0.4053	0.3983	0.3441	1.776	0.7847	1.308
						0.6	0.4451	0.4373	0.2374	1.749	0.5115	1.279
						1.0	0.5026	0.4934	0.0	2.692	0.0	2.227
	-0.4	1	0	--	0.7	0.15	0.3149	0.2941	0.4182	1.540	1.089	1.281
						0.4	0.4247	0.3863	0.3193	1.467	0.7311	1.219
						0.6	0.4807	0.4328	0.2182	1.416	0.4710	1.177
						1.0	0.5619	0.5001	0.0	2.338	0.0	2.114

Table 2, cont.

ω	f_w	t_s	σ	$\left(\frac{u_e}{u_\infty}\right)^2$	Pr	t_w	$g'(0)$	$\theta'(0)$	I_2	$I_1(1)$	$I_1(2)$	$I_1(3)$
0.7	-0.4	1	0.5	--	0.7	0.15	0.3031	0.2908	0.4255	1.644	1.107	1.303
						0.4	0.4025	0.3810	0.3252	1.582	0.7445	1.241
						0.6	0.4529	0.4265	0.2232	1.535	0.4798	1.200
						1.0	0.5258	0.4923	0.0	2.460	0.0	2.136
			0.9	--	0.7	0.15	0.2683	0.2653	0.4844	1.960	1.257	1.478
						0.4	0.3469	0.3421	0.3727	1.918	0.8523	1.420
						0.6	0.3865	0.3809	0.2553	1.874	0.5510	1.377
						1.0	0.4437	0.4368	0.0	2.797	0.0	2.309
1	0	0.1	1	1	1	0	0.9076	0.9076	-1.115	0.9629	-1.841	0.6826
						0.5	1.189	1.189	-2.871	0.7798	-4.278	0.5481
						1.0	1.339	1.339	-4.528	0.7030	-6.327	0.4929
		0.1538	1	1	1	0	0.8258	0.8258	-0.5459	1.051	-0.9319	0.7461
						0.5	1.075	0.075	-1.736	0.8575	-2.756	0.6034
						1.0	1.208	1.208	-2.885	0.7753	-4.265	0.5442
		0.3333	1	1	1	0	0.7063	0.7063	0.0364	1.210	0.1645	0.8615
						0.5	0.9028	0.9028	-0.5265	1.006	-0.9472	0.7098
						1.0	1.010	1.010	-1.104	0.9161	-1.832	0.6445
		0.625	1	1	1	0	0.6326	0.6326	0.2647	1.331	0.7220	0.9504
						0.5	0.7915	0.7915	-0.0202	1.130	-0.0382	0.7994
						1.0	0.8793	0.8793	-0.3317	1.037	-0.6223	0.7317
		1	1	--	1	0	0.5898	0.5898	0.3567	1.412	1.010	1.010
						0.5	0.7230	0.7230	0.1965	1.220	0.4327	0.8654
						1.0	0.7979	0.7979	0.0	1.128	0.0	0.7979

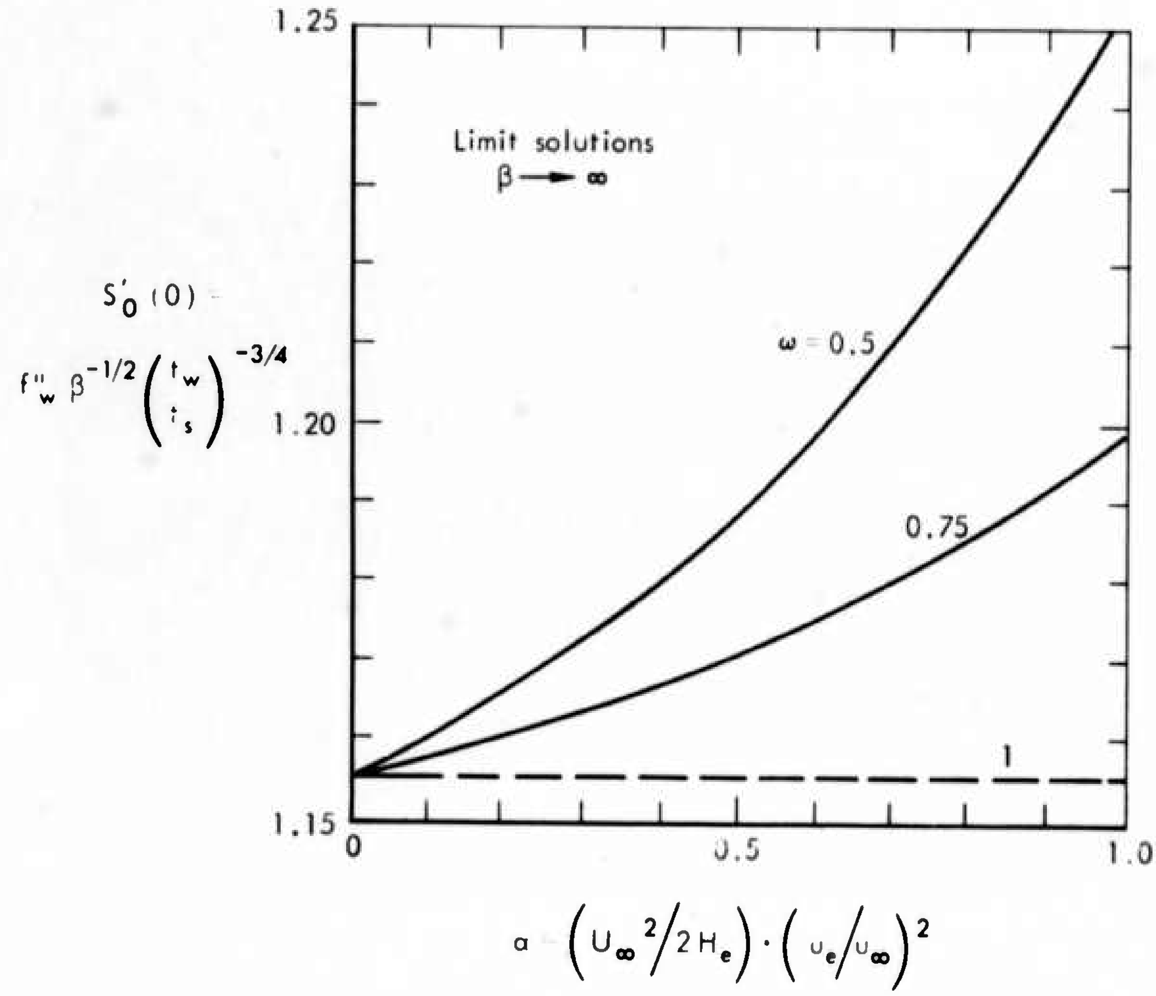

Fig. 1—Inner solutions for $\beta \rightarrow \infty$

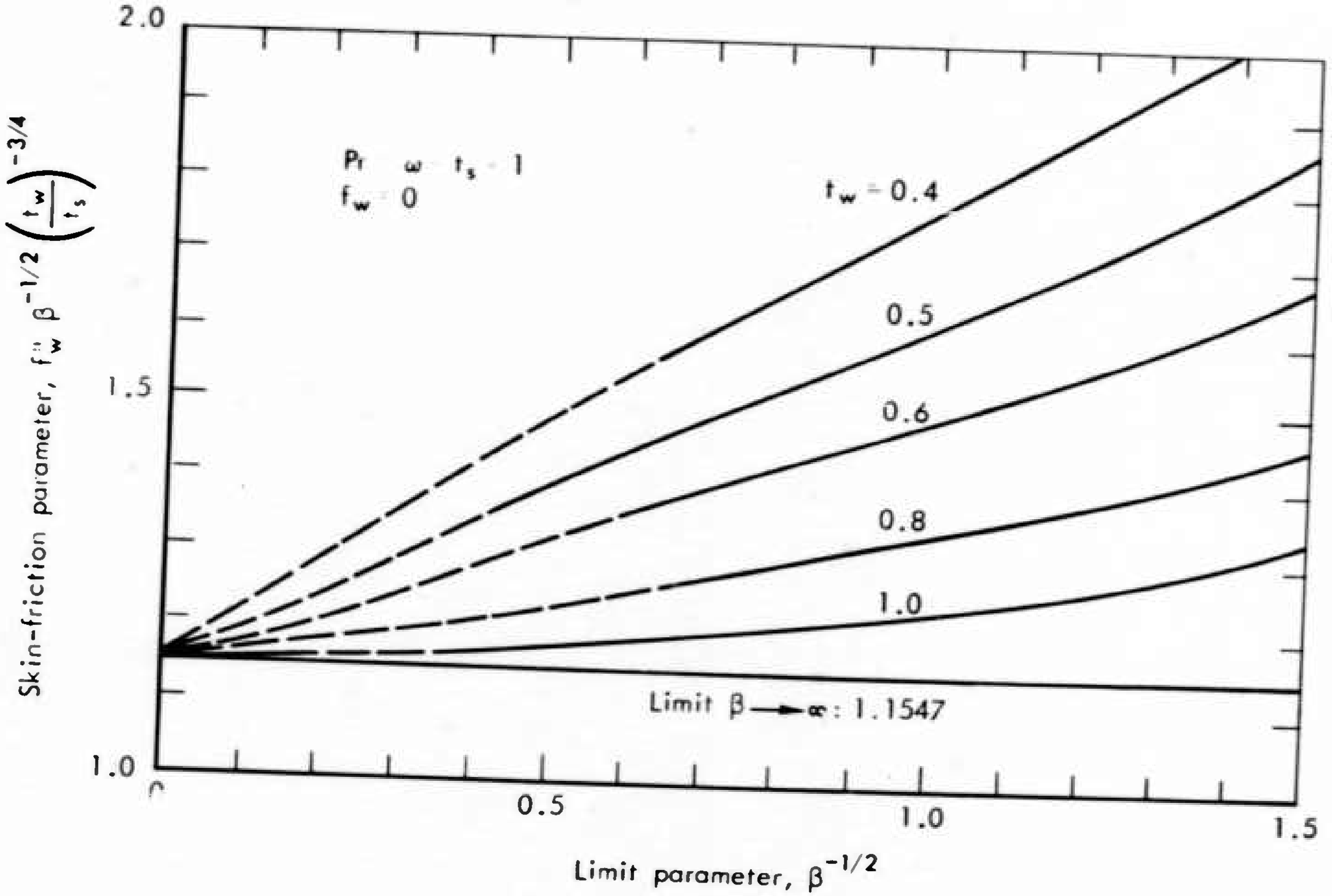

Fig. 2—Variation of skin friction with wall temperature for large β

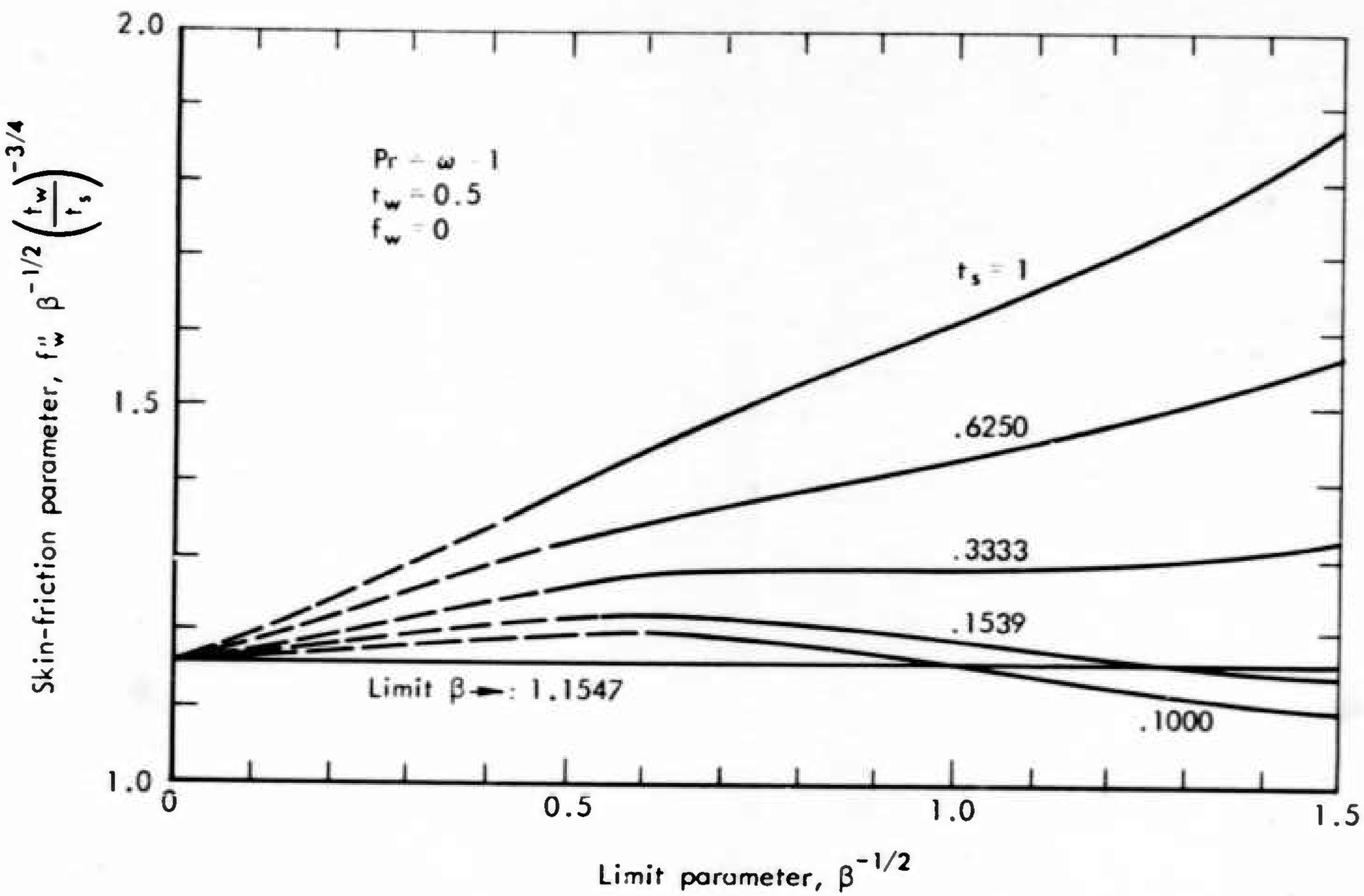

Fig.3—Variation of skin friction with t_s for large β

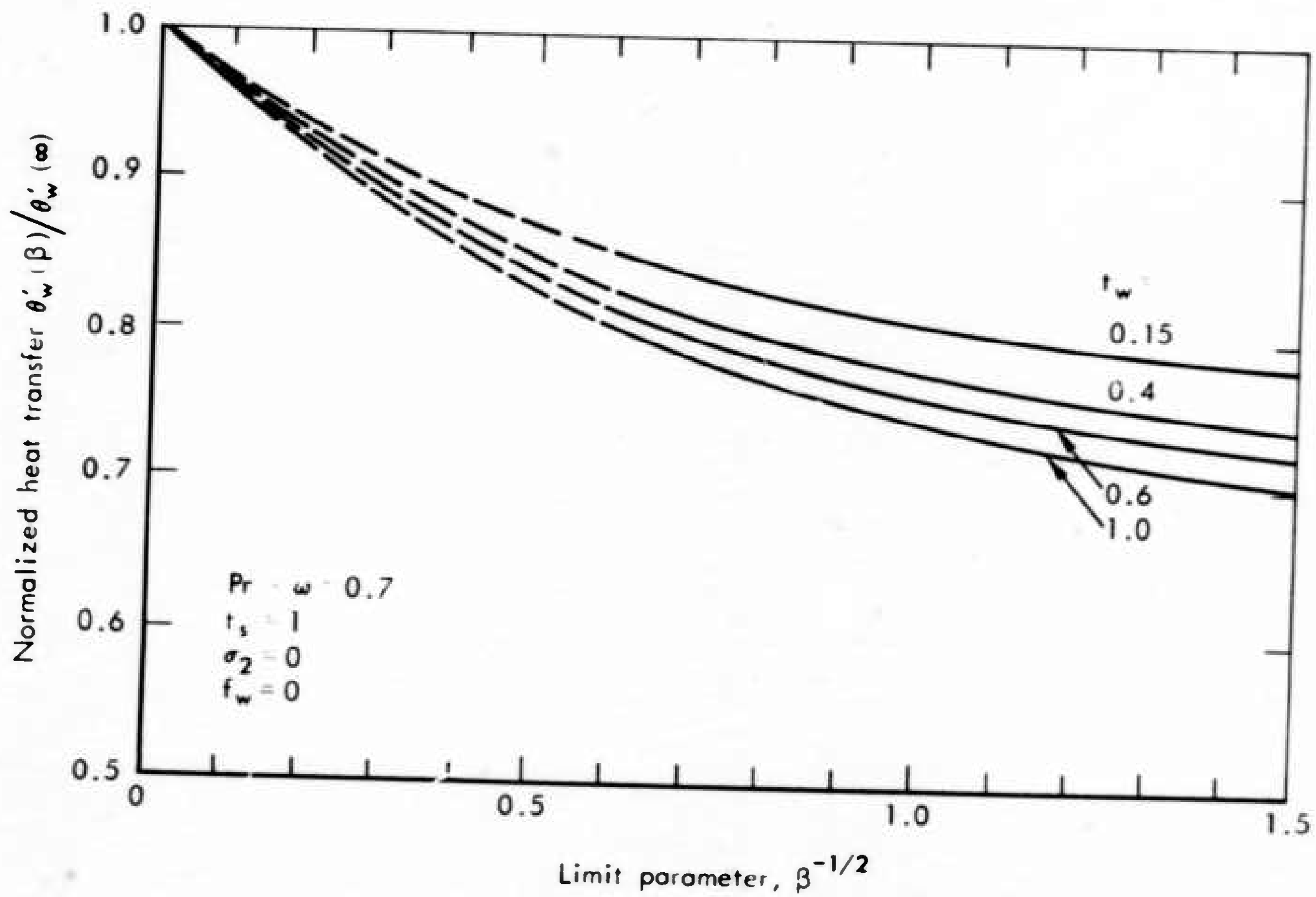

Fig.4—Variation of heat transfer with wall temperature for large β

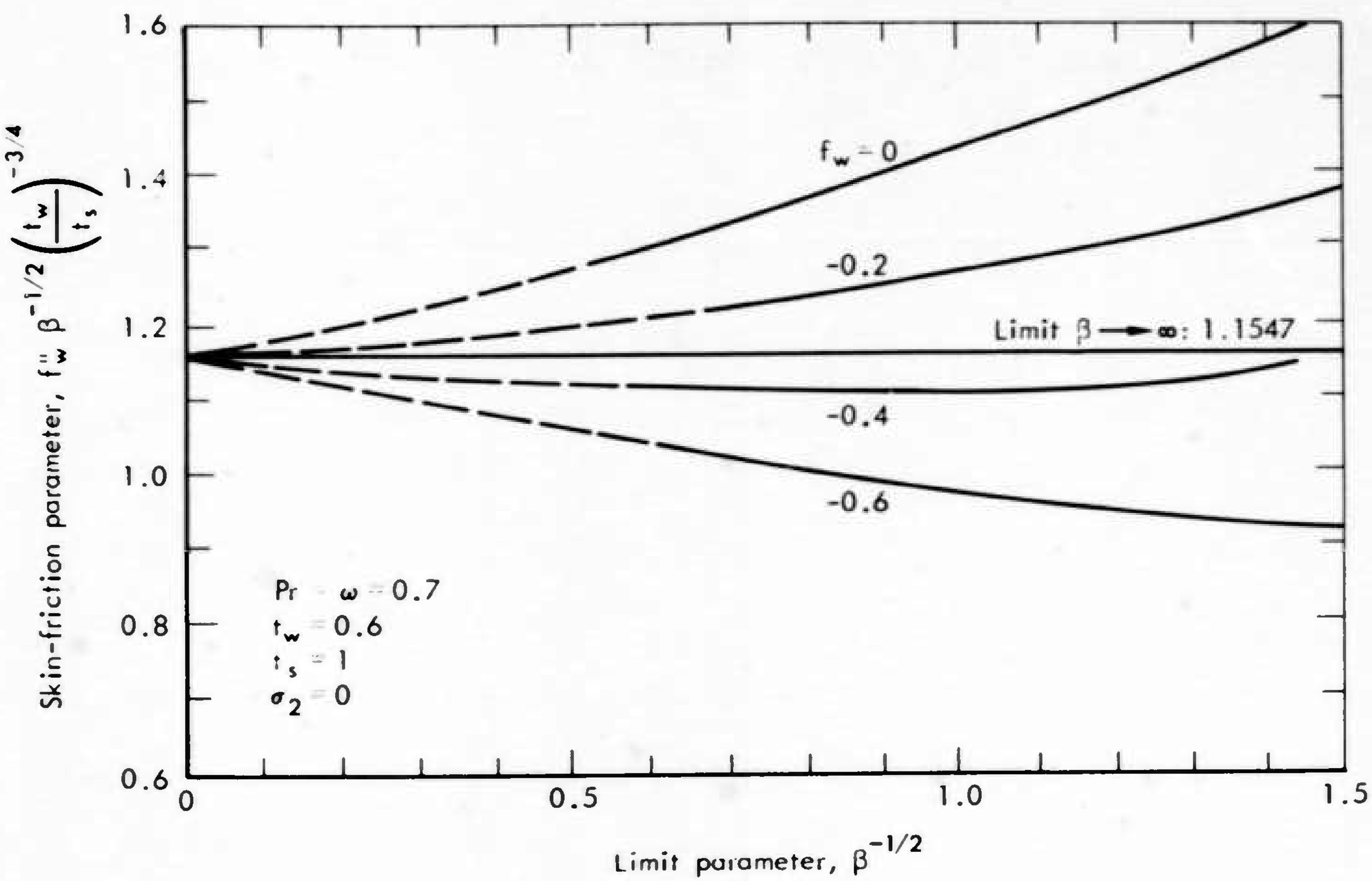

Fig.5—Variation of skin friction with injection for large β

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